Connection Formula for Airy-type Equations(Painleve Transcendents and Asymptotic Analysis)

AUTHOR(S):

Ohyama, Yousuke

CITATION:

Ohyama, Yousuke. Connection Formula for Airy-type Equations(Painleve Transcendents and Asymptotic Analysis). 数理解析研究所講究録 1995, 931: 1-19

ISSUE DATE:

1995-12

URL:

http://hdl.handle.net/2433/59962

数理解析研究所講究録
931 巻 1995 年 1-19

Connection Formula for Airy-type Equations

Yousuke Ohyama

大山 陽介

(大阪大. 理)

§0. Introduction

We will study turning point problems for third order equations using technique of WKB analysis. In the case of second order equations, there are many results which go back to Liouville. It seems that there are some difficulty to generalize higher-order case ([BNR], [AKT2]). We treat one of the most simple example in this paper.

In 1936, H.Scheffé study the equations

$$\frac{d^m y}{dz^m} - z^n y = 0, \tag{0.1}$$

which he called t-equation. He showed that the asymptotic forms of t-equation play a key role in the study of more general equations ([S]). The asymptotic behavior of t-equation is studied by Turrittin ([T]). He calculated Stokes multipliers of t-equation around infinity.

The Stokes sectors of t-equation are divided by $2(n+m)$ lines

$$\arg z = \frac{\pi h}{n+m}, \qquad h = 0, 1, \cdots, 2(n+m)-1.$$

But different solutions may have the same asymptotic expansions in some sectors. Hence we should choose special solutions in order to determine the Stokes multipliers. Turrittin used Barnes-type integral formula of solutions, and choose an enlarged sector to restore uniqueness.

Recently Silverstone and others studied the resonances of LoSurdo-Stark effects and the energy eigenvalue of $H_2{}^+$. In [SNH], they consider Borel summablity of Airy function $\mathrm{Ai}(z)$. The asymptotic expansion of $\mathrm{Ai}(z)$

$$\mathrm{Ai}(z) \sim \frac{1}{2}\pi^{-\frac{1}{2}} z^{-\frac{1}{4}} \exp\left(-\frac{2}{3}z^{\frac{3}{2}}\right) \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma\left(k+\frac{5}{6}\right)\Gamma\left(k+\frac{1}{6}\right)}{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{6}\right)2^k k!} \left(-\frac{2}{3}z^{\frac{3}{2}}\right)^k$$

is valid for $|\arg z| < \pi$. If we take the Borel resummation of the expansion, we have

$$\mathrm{Ai}(z) = \frac{1}{2}\pi^{-\frac{1}{2}} z^{-\frac{1}{4}} \exp\left(-\frac{2}{3}z^{\frac{3}{2}}\right) \int_0^{\infty} e^{-t} {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; -\frac{3t}{2z^{\frac{3}{2}}}\right) dt.$$

By looking at the branch points of the hypergeometric function, they show that the standard domain of Borel summability is $|\arg z| < \frac{2}{3}\pi$, narrower than the usual sector $|\arg z| < \pi$. Borel resummation gives an integral expression of asymptotic expansions, and we can understand the valid sectors i from the branch points of the integrand.

On the other hand, the exact WKB method invented by A.Voros gives a powerful tool in the study of semi-classical analysis of one-dimensional Shrödinger equation ([V]). T.Aoki, T.Kawai and Y.Takei developed the Voros theory, and give exact connection formula of second order equations with regular singularity. In [AKT], they reduce general equations to the Airy equation near a simple turning point and give the connection formula near turning point. They also study the case of third order equations ([AKT2]). In their theory the connection problem of the Airy equation is essential and this problem is just the same as the case of Silverstone.

We will study the connection problem of the Airy-type equation (0.1) when $m = 3$. The exact WKB analysis of Voros is also useful in this case. In section 1 we review exact WKB analysis. We must modify Voros's theory for the third order equations. In section 2 we discuss WKB solutions of Airy-type equation. The Borel transformations of WKB solutions to this type of equation can be represented by generalized hypergeometric functions ${}_3F_2$. We will give an explicit connection formula of Borel-transform of WKB solutions. In section 3, using the connection formula of ${}_3F_2$, we study connection problem of generalized Airy equation around infinity. In section 4, we study connection problem from infinity to zero. In both cases, the connection formula of hypergeometric functions ${}_2F_1$ plays a key role.

§1. Complex WKB analysis

We first review complex WKB analysis. We follow the notation in [V] and [AKT], although they discuss second order equations.

We are concerned with an equation of the following form:

$$(1.1) \qquad (\frac{d^3}{dq^3} - x^3 Q(q))\psi(q,x) = 0$$

where $Q(q)$ is an analytic function and x is a complex large parameter. We take a formal solution of the following type:

$$(1.2) \qquad \psi(q,x) = \frac{1}{\sqrt{x}} \exp(\int^q S(q',x)dq'),$$

where

$$(1.3) \qquad S(q,x) = S_{-1}(q)x + S_0(q) + S_0(q)x^{-1} + \cdots .$$

As a formal power series in x^{-1}, $S(q,x)$ satisfies following nonlinear equation:

$$3S(q,x)\frac{d}{dq}S(q,x)+\frac{d^2}{dq^2}S(q,x)+S(q,x)^3-x^3Q(q)=0. \tag{1.4}$$

Each term $S_j(q)$ in (1.3) is uniquely determined if we fix the branch of $S_{-1}=\sqrt[3]{Q(q)}$. From (1.4) $S_j(q)$ is calculated by the following recursive equation:

$$\begin{aligned}3S_{-1}(q)^2S_{m+2}+&\frac{d^2}{dq^2}S_m+3\sum_{i+j=m}S_i\frac{d}{dq}S_j\\ &+3S_{-1}\sum_{i+j=m+1,i,j\geq 0}S_iS_j+\sum_{i+j+k=m,i,j,k\geq 0}S_iS_jS_k=0.\end{aligned}$$

Let

$$S_{(0)}=\sum_j S_{3j}x^{3j},\ S_{(1)}=\sum_j S_{3j+1}x^{3j+1},\ S_{(2)}=\sum_j S_{3j+2}x^{3j+2}.$$

By (4) we have

$$3\frac{d}{dq}\left(S_{(1)}S_{(0)}\right)+3S_{(2)}\frac{d}{dq}S_{(2)}+\frac{d^2}{dq^2}S_{(1)}+3S_{(0)}^2S_{(1)}+3S_{(1)}^2S_{(2)}+3S_{(2)}^2S_{(0)}=0,$$

$$3\frac{d}{dq}\left(S_{(2)}S_{(0)}\right)+3S_{(1)}\frac{d}{dq}S_{(1)}+\frac{d^2}{dq^2}S_{(2)}+3S_{(0)}^2S_{(2)}+3S_{(1)}^2S_{(0)}+3S_{(2)}^2S_{(1)}=0.$$

Taking the difference of the equations above,

$$\begin{aligned}S_{(2)}\frac{d^2}{dq^2}S_{(1)}-S_{(1)}\frac{d^2}{dq^2}S_{(2)}+3S_{(2)}^2\frac{d}{dq}S_{(2)}-3S_{(1)}^2\frac{d}{dq}S_{(1)}+3S_{(2)}\frac{d}{dq}\left(S_{(1)}S_{(0)}\right)\\ -3S_{(1)}\frac{d}{dq}\left(S_{(2)}S_{(0)}\right)-3S_{(0)}\left(S_{(2)}^3-S_{(1)}^3\right)=0.\end{aligned}$$

Hence

$$\begin{aligned}S_{(0)}&\left(S_{(2)}^3-S_{(1)}^3+S_{(2)}\frac{d}{dq}S_{(1)}-S_{(1)}\frac{d}{dq}S_{(2)}\right)\\ &+\frac{1}{3}\left(3S_{(2)}^2\frac{d}{dq}S_{(2)}-3S_{(1)}^2\frac{d}{dq}S_{(1)}S_{(2)}\frac{d^2}{dq^2}S_{(1)}-S_{(1)}\frac{d^2}{dq^2}S_{(2)}\right)=0.\end{aligned}$$

Therefore we obtain

$$S_{(0)}=-\frac{1}{3}\frac{d}{dq}\log\left(S_{(2)}^3-S_{(1)}^3+S_{(2)}\frac{d}{dq}S_{(1)}-S_{(1)}\frac{d}{dq}S_{(2)}\right).$$

Therefore WKB solutions may be written in the following form:

$$\psi(q,x)=\frac{\sqrt{x}}{\sqrt[3]{S_{(2)}^3-S_{(1)}^3+S_{(2)}\frac{d}{dq}S_{(1)}-S_{(1)}\frac{d}{dq}S_{(2)}}}\exp\int(S_{(1)}+S_{(2)})dq', \tag{1.4}$$

Although the series (1.3) is divergent, Voros shows that the Borel transform with respect to the parameter x is a ramified analytic function. We will fix the definition of the Borel transform (see [V], [AKT]):

DEFINITION. *Let $f(x)$ be a formal series*

$$f(x)=e^{\xi_0 x}\sum_{j\geq 0} f_j x^{-j-a},$$

where a is any complex number. Then its Borel transform $f_B(\xi)$ is defined by the series

$$\sum_{j\geq 0}\frac{f_j}{\Gamma(j+a+1)}(\xi+\xi_0)^{j+a}.$$

At least formally, we can represent $f(x)$ as an integral of $f_B(\xi)$:

$$f(x)=x\int_{-\xi_0}^{\infty} e^{-x\xi} f_B(\xi)d\xi. \tag{1.5}$$

If $\psi(q,x)$ is a WKB solution of (1.1), it follows from (1.5) that $\psi_B(\xi,q)$ satisfies the Balian-Bloch equation

$$\frac{d^3}{dq^3}\psi_B(\xi,q)-Q(q)\frac{d^3}{d\xi^3}\psi_B(\xi,q)=0. \tag{1.6}$$

§2. WKB solutions of Airy-type equation

In [AKT], it is shown that the Borel transformed WKB solutions of Airy equation are represented by Gauss hypergeometric functions. The connection formula is calculated using Kummer's relation for hypergeometric functions. In this section we study following Airy-type equation:

$$\frac{d^3}{dq^3}\psi - x^3 q^n\psi=0. \tag{2.1}$$

The WKB solution of (2.1) is as follows:

$$\psi(q,x)=\frac{1}{\sqrt{x}}\exp\int^q S(x,q')dq', \tag{2.2}$$

where

$$S(q,x)=S_{-1}(q)x+S_0(q)+S_0(q)x^{-1}+\cdots. \tag{2.3}$$

Let $\psi^{(0)}(q,x),\psi^{(1)}(q,x)$ and $\psi^{(-1)}(q,x)$ be the WKB solutions corresponding to the initial term $S_{-1}=q^{\frac{n}{3}},q^{\frac{n}{3}}\omega$ and $q^{\frac{n}{3}}\omega^{-1}$, respectively, where $\omega=e^{\frac{2}{3}\pi i}$. We take a branch of $q^{n/3}$ so that $\Re q^{n/3} > 0$ if $q > 0$. We can verify by the induction that the $S_k(q)$ has the form

$$S_k(q) = s_k q^{-1-(n+3)\frac{k}{3}},$$

where s_k is a complex number. For example, if $S_{-1}(q) = \omega^j q^{n/3}$,

$$S_0(q) = -\frac{n}{3}q^{-1}, S_1(q) = -\frac{n(6+n)\omega^2 j}{27}q^{-\frac{n+6}{3}}, S_{-2}(q) = -\frac{(18n+9n^2+n^3)\omega^j}{81}q^{-\frac{2n+9}{3}}\ldots.$$

Therefore $\psi^{(j)}(q,x)$ is following:

$$\psi^{(j)}(q,x) = \exp(\frac{3\omega^j}{n+3}q^{\frac{n+3}{3}}x)q^{-\frac{n}{3}}\sum_{k\geq 0} T_k(q)x^{-k-\frac{1}{2}}, \tag{2.4}$$

and $T_k(q)$ has also the form

$$T_k(q) = t_k q^{-(n+3)\frac{k}{3}},$$

where t_k is a complex number. Especially $t_0 = 1$. We denote the Borel transform of $\psi^{(j)}(q,x)$ by $\psi_B^{(j)}(\xi,q)$. By definition, we have

$$\begin{aligned}
\psi_B^{(j)}(\xi,q) &= q^{-\frac{n}{3}}\sum_{k\geq 0}\frac{t_j q^{-(n+3)k/3}}{\Gamma(k+\frac{3}{2})}(\xi + \frac{3\omega^j}{n+3}q^{\frac{n+3}{3}})^{k+\frac{1}{2}} \\
&= q^{\frac{3-n}{6}}\sum_{k\geq 0}\frac{t_k}{\Gamma(k+\frac{3}{2})}(\xi q^{-\frac{n+3}{3}} + \frac{3\omega^j}{n+3})^{k+\frac{1}{2}} \\
&= q^{\frac{3-n}{6}}\frac{2}{\sqrt{\pi}}\sum_{k\geq 0}\frac{t_k}{(\frac{3}{2})_k}(\xi q^{-\frac{n+3}{3}} + \frac{3\omega^j}{n+3})^{k+\frac{1}{2}},
\end{aligned} \tag{2.5}$$

where $(a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)}$.

Let t denote $\xi q^{-\frac{n+3}{3}}$. Then $\psi_B^{(j)}(\xi,q)$ has the form

$$\psi_B^{(j)}(\xi,q) = q^{\frac{3-n}{6}}h^{(j)}(t).$$

$h^{(j)}(t)$ has a singularity at $t = -\dfrac{3\omega^j}{n+3}$. It follows from (1.6) that $y = h^{(0)}(t)$, $h^{(1)}(t)$, $h^{(-1)}(t)$ satisfy the equation

$$\begin{aligned}
&((n+3)^3t^3+27)\frac{d^3y}{dt^3} + \frac{9(3+n)^3}{2}t^2\frac{d^2y}{dt^2} \\
&+ \frac{(3+n)(81+78n+13n^2)}{4}t\frac{dy}{dt} + \frac{(-3+n)(3+n)(9+n)}{8}y = 0.
\end{aligned} \tag{2.6}$$

The three regular singular points in (2.6) correspond to the choice of S_{-1}. The exponents at each singularity are $0, \frac{1}{2}, 1$. Although the distance of the exponents may be an integer, any local solution does not have a logarithmic term. The exponent of $h^{(j)}(t)$ is $\frac{1}{2}$ at the corresponding singular point. We consider analytic continuation of $h^{(j)}(t)$ to another singular point.

The equation (2.6) is invariant by the $\frac{2}{3}\pi$-rotation around the origin. Let s denote $-\frac{(n+3)^3}{27}t^3$. Then we obtain

$$
\begin{aligned}
&s^2(s-1)\frac{d^3y}{ds^3}-(2s-\frac{7}{2}s^2)\frac{d^2y}{ds^2}\\
&\qquad -(\frac{2}{9}-\frac{159+114n+19n^2}{12(3+n)^2}s)\frac{dy}{ds}+\frac{(-3+n)(9+n)}{216(3+n)^2}y=0
\end{aligned}
\tag{2.7}
$$

We set $p=\frac{1}{n+3}$. The local solutions of (2.7) at origin are followings:

$$
\begin{aligned}
\phi_0^{(0)}(s)&= {}_3F_2(\frac{1}{6}-\frac{p}{2},\frac{1}{6}+\frac{p}{2},\frac{1}{6};\frac{1}{3},\frac{2}{3};s),\\
\phi_0^{(1)}(s)&=s^{\frac{1}{3}}{}_3F_2(\frac{1}{2}-\frac{p}{2},\frac{1}{2}+\frac{p}{2},\frac{1}{2};\frac{2}{3},\frac{4}{3};s),\\
\phi_0^{(2)}(s)&=s^{\frac{2}{3}}{}_3F_2(\frac{5}{6}-\frac{p}{2},\frac{5}{6}+\frac{p}{2},\frac{5}{6};\frac{4}{3},\frac{5}{3};s).
\end{aligned}
$$

We set

$$
\begin{aligned}
u_1(s)&= {}_2F_1(\frac{1}{12}-\frac{p}{2},\frac{1}{12}+\frac{p}{2};\frac{2}{3};s),\\
u_2(s)&= s^{\frac{1}{3}}{}_2F_1(\frac{5}{12}-\frac{p}{2},\frac{5}{12}+\frac{p}{2};\frac{4}{3};s).
\end{aligned}
$$

Using Clausen's formula

$$
{}_3F_2(2a,a+b,2b;2a+2b,a+b+\frac{1}{2};s)=[{}_2F_1(a,b;a+b+\frac{1}{2};s)]^2,
$$

$$
\begin{aligned}
&{}_3F_2(\frac{1}{2},a-b+\frac{1}{2},-a+b+\frac{1}{2};-a-b+\frac{3}{2},a+b+\frac{1}{2};s)\\
&\qquad={}_2F_1(a,b;a+b+\frac{1}{2};s)\,{}_2F_1(\frac{1}{2}-a,\frac{1}{2}-b;-a-b+\frac{3}{2};s),
\end{aligned}
$$

we have

$$
\begin{aligned}
\phi_0^{(0)}(s)&=u_1(s)^2,\\
\phi_0^{(1)}(s)&=u_1(s)u_2(s),\\
\phi_0^{(2)}(s)&=u_2(s)^2.
\end{aligned}
$$

We set

$$u_3(s) = \quad {}_2F_1\left(\frac{1}{12}-\frac{p}{2},\frac{1}{12}+\frac{p}{2};\frac{1}{2};1-s\right),$$
$$u_4(s) = (1-s)^{1/2}{}_2F_1\left(\frac{7}{12}-\frac{p}{2},\frac{7}{12}+\frac{p}{2};\frac{3}{2};1-s\right).$$

Then the products

$$\phi_1^{(0)}(s) = u_3(s)^2,$$
$$\phi_1^{(1)}(s) = u_3(s)u_4(s),$$
$$\phi_1^{(2)}(s) = u_4(s)^2.$$

are independent solutions of (2.7) at $s = 1$. Since $\phi_1^{(1)}(s)$ has a singularity of the type $(1-s)^{1/2}$, $h^{(j)}(t)$ is equal to $\phi_1^{(1)}\left(-\left(\frac{t}{3p}\right)^3\right)$ up to constant multiplication. We notice the following lemma.

LEMMA. *Consider the functions*

$$\mu(t) = \left(1+\left(\frac{t}{3p}\right)^3\right)^{\frac{1}{2}}, \qquad \mu_j(t) = \left(t+3p\omega^j\right)^{\frac{1}{2}}. \qquad (j=-1,0,1)$$

We take the following cut lines in t-plane:

$$\left\{t;\ t+\frac{\omega^j}{3p}\in\mathbb{R}_+\right\}. \qquad (j=-1,0,1)$$

We take a branch of $\mu(t)$ and $\mu_j(t)$ as follows.

$$\lim_{\varepsilon\downarrow 0}\mu(a+i\varepsilon) > 0,$$
$$\lim_{\varepsilon\downarrow 0}\mu(a-\frac{\omega^j}{3p}+i\varepsilon) > 0,$$

for $a > 0$. Then we can take functions $A_j(t)$ $(j=-1,0,1)$, which are holomorphic at $t = -\dfrac{\omega^j}{3p}$ such that

$$\mu(t) = \mu_j(t)A_j(t),$$

and $A_j\left(-\dfrac{\omega^j}{3p}\right) = \dfrac{\omega^j}{\sqrt{p}}$ for $j=-1,0,1$.

The lemma is easily verified by direct calculations.

By (2.5), $h^{(j)}(t)$ has a singularity at $t = -\dfrac{\omega^j}{3p}$ such that

$$h^{(j)}(t) = \frac{2}{\sqrt{\pi}}(t+3p\omega^j)^{\frac{1}{2}}\left(1+O\left(t+3\omega^j p\right)\right).$$

Therefore it follows from the lemma above that

$$h^{(j)}(t) = 2\omega^{-j}\sqrt{\frac{p}{\pi}}\phi_1^{(1)}\left(-\left(\frac{t}{3p}\right)^3\right), \tag{2.8}$$

near $t = -3p\omega^j$.

We will calculate the discontinuity of $h^{(0)}(t)$ at $t = -3p\omega, -3p\omega^{-1}$. In the s-space we should consider the analytic continuation around the origin. We denote $\tilde{\phi}_1^{(j)}(s)$ (resp., $\tilde{\tilde{\phi}}_1^{(j)}(s)$) as the analytic continuation of $\phi_1^{(j)}(s)$ once counter-clockwise (resp., clockwise) around the origin.

PROPOSITION 2.1. *We have*

$$\tilde{\phi}_1^{(1)}(s) = \left(2\cos^2 p\pi - \frac{1}{2}\right)\omega\phi_1^{(1)}(s) + (\textit{holomorphic at } \ s = 1),$$
$$\tilde{\tilde{\phi}}_1^{(1)}(s) = \left(2\cos^2 p\pi - \frac{1}{2}\right)\omega^2\phi_1^{(1)}(s) + (\textit{holomorphic at } \ s = 1),$$

near $s = 1$.

PROOF: By Gauss' connection formula

$$\begin{aligned}{}_2F_1(a,b;c;s) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1(a,b;a+b-c+1;1-s)\\ &+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-s)^{c-a-b}{}_2F_1(c-a,c-b;c-a-b+1;1-s),\end{aligned}$$

we have

$$\begin{aligned} u_1(s) &= Au_3(s) - Bu_4(s),\\ s^{1/3}u_2(s) &= Cu_3(s) - Du_4(s),\end{aligned} \tag{2.9}$$

where

$$A = \frac{\sqrt{\pi}\Gamma(\frac{2}{3})}{\Gamma(\frac{7}{12}+\frac{p}{2})\Gamma(\frac{7}{12}-\frac{p}{2})}, \quad B = \frac{2\sqrt{\pi}\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{12}+\frac{p}{2})\Gamma(\frac{1}{12}+\frac{p}{2})},$$
$$C = \frac{\sqrt{\pi}\Gamma(\frac{3}{4})}{\Gamma(\frac{11}{12}+\frac{p}{2})\Gamma(\frac{11}{12}+\frac{p}{2})}, \quad D = \frac{2\sqrt{\pi}\Gamma(\frac{4}{3})}{\Gamma(\frac{5}{12}+\frac{p}{2})\Gamma(\frac{5}{12}+\frac{p}{2})}.$$

Therefore

$$\begin{pmatrix}\phi_0^{(0)}(s)\\ \phi_0^{(1)}(s)\\ \phi_0^{(2)}(s)\end{pmatrix} = \begin{pmatrix} A^2 & -2AB & B^2\\ AC & -(AD+BC) & BD\\ C^2 & -2CD & D^2\end{pmatrix}\begin{pmatrix}\phi_1^{(0)}(s)\\ \phi_1^{(1)}(s)\\ \phi_1^{(2)}(s)\end{pmatrix}. \tag{2.10}$$

We denote $\tilde{\phi}_1^{(j)}(s)$ (resp., $\tilde{\tilde{\phi}}_1^{(j)}(s)$) as the analytic continuation of $\phi_0^{(j)}(s)$ once counter-clockwise (resp., clockwise) around the origin. Then we have

$$
\begin{aligned}
\tilde{\phi}_0^{(j)}(s) &= \omega^j \phi_0^{(j)}(s), \\
\tilde{\tilde{\phi}}_0^{(j)}(s) &= \omega^{2j} \phi_0^{(j)}(s).
\end{aligned} \tag{2.11}
$$

By direct calculation, we have

$$
AD = \frac{1}{3} + \frac{2}{3\sqrt{3}} \cos \pi p, \quad BC = -\frac{1}{3} + \frac{2}{3\sqrt{3}} \cos \pi p.
$$

Hence the inverse transformation of (2.9) is given by

$$
\begin{aligned}
u_3(s) &= \frac{3}{2} D u_1(s) - \frac{3}{2} B u_2(s), \\
u_4(s) &= \frac{3}{2} C u_1(s) - \frac{3}{2} A u_2(s).
\end{aligned}
$$

Therefore

$$
\begin{pmatrix} \phi_1^{(0)}(s) \\ \phi_1^{(1)}(s) \\ \phi_1^{(2)}(s) \end{pmatrix} = \frac{9}{4} \begin{pmatrix} D^2 & -2BD & B^2 \\ CD & -(AD+BC) & AB \\ C^2 & -2AC & A^2 \end{pmatrix} \begin{pmatrix} \phi_0^{(0)}(s) \\ \phi_0^{(1)}(s) \\ \phi_0^{(2)}(s) \end{pmatrix} \tag{2.12}
$$

(2.12) is valid if we change all of $\phi_k^{(j)}$ into $\tilde{\phi}_k^{(j)}$ or $\tilde{\tilde{\phi}}_k^{(j)}$. Combined with (2.10), (2.11) and (2.12), we obtain the proposition 2.1. ∎

From now on, we will study the analytic continuation from $s = \infty$ to $s = 1$. We set

$$
\begin{aligned}
u_5(s) &= (-s)^{-\frac{1}{12}+\frac{p}{2}} {}_2F_1\left(\frac{1}{12} - \frac{p}{2}, \frac{5}{12} - \frac{p}{2}; 1-p; s^{-1}\right), \\
u_6(s) &= (-s)^{-\frac{1}{12}-\frac{p}{2}} {}_2F_1\left(\frac{1}{12} + \frac{p}{2}, \frac{5}{12} + \frac{p}{2}; 1+p; s^{-1}\right).
\end{aligned}
$$

The products

$$
\begin{aligned}
\phi_\infty^{(0)}(s) &= u_5(s)^2, \\
\phi_\infty^{(1)}(s) &= u_5(s) u_6(s), \\
\phi_\infty^{(2)}(s) &= u_6(s)^2.
\end{aligned}
$$

are the local solutions of (2.7) near infinity. By Clausen's formula we have

$$
\begin{aligned}
\phi_\infty^{(0)}(s) &= (-s)^{-\frac{1}{6}+p} \, {}_3F_2\left(\frac{1}{6} - p, \frac{1}{2} - p, \frac{5}{6} - p; 1-p, 1-2p; s^{-1}\right), \\
\phi_\infty^{(1)}(s) &= (-s)^{-\frac{1}{6}} \, {}_3F_2\left(\frac{1}{6}, \frac{1}{2}, \frac{5}{6}; 1-p, 1+p; s^{-1}\right), \\
\phi_\infty^{(2)}(s) &= (-s)^{-\frac{1}{6}-p} \, {}_3F_2\left(\frac{1}{6} + p, \frac{1}{2} + p, \frac{5}{6} + p; 1+p, 1+2p; s^{-1}\right).
\end{aligned}
$$

Proposition 2.2. *The analytic continuation of $\phi_\infty^{(k)}(s)$ along a path in the lower-half plane is*

$$\phi_\infty^{(k)}(s) = -\frac{3^{3(k-1)p}}{\sqrt{\pi}} e^{i\pi(-\frac{2}{3}+(1-k)p)} \frac{\prod_{j=0}^{2} \Gamma(1+(k-j)p)}{\Gamma\left(\frac{1}{2}+3(k-1)p\right)} \phi_1^{(1)}(s) + (\textit{holomorphic at } \ s=1)$$

near $s=1$.

Proof: When we take a path in the lower-half plane, the connection formula is the following:

$$\begin{aligned} &(-s)^{-a}{}_2F_1(a, a+1-c; a+1-b; s^{-1}) \\ =&\frac{\Gamma(c+1-a-b)\Gamma(a+b-c)\Gamma(a+1-b)}{\Gamma(1-b)\Gamma(c-b)\Gamma(a+b+1-c)} e^{-i\pi a}{}_2F_1(a,b;a+b-c+1;1-s) \\ &-\frac{\Gamma(a+b-c)\Gamma(a+1-b)}{\Gamma(a)\Gamma(a+1-c)} e^{i\pi(b-c)}(1-s)^{c-a-b}{}_2F_1(c-a,c-b;c-a-b+1;1-s). \end{aligned}$$

Therefore

$$\begin{aligned} u_5(s) &= Eu_3(s) + Fu_4(s), \\ u_6(s) &= Gu_3(s) + Hu_4(s), \end{aligned} \tag{2.13}$$

where

$$E = -e^{i\pi(-\frac{1}{12}+\frac{p}{2})} \frac{\sqrt{\pi}\Gamma(1-p)}{\Gamma\left(\frac{11}{12}-\frac{p}{2}\right)\Gamma\left(\frac{7}{12}-\frac{p}{2}\right)}, \quad F = e^{i\pi(-\frac{7}{12}+\frac{p}{2})} \frac{2\sqrt{\pi}\Gamma(1-p)}{\Gamma\left(\frac{1}{12}-\frac{p}{2}\right)\Gamma\left(\frac{5}{12}-\frac{p}{2}\right)},$$

$$G = -e^{i\pi(-\frac{1}{12}-\frac{p}{2})} \frac{\sqrt{\pi}\Gamma(1+p)}{\Gamma\left(\frac{11}{12}+\frac{p}{2}\right)\Gamma\left(\frac{7}{12}+\frac{p}{2}\right)}, \quad H = e^{i\pi(-\frac{7}{12}-\frac{p}{2})} \frac{2\sqrt{\pi}\Gamma(1+p)}{\Gamma\left(\frac{1}{12}+\frac{p}{2}\right)\Gamma\left(\frac{5}{12}+\frac{p}{2}\right)}.$$

At first we consider $\phi_\infty^{(0)}(s)$. By (2.13) we have

$$u_5(s)^2 = 2GHu_3(s)u_4(s) + (\text{holomorphic at } \ s=1).$$

Recall the multiplication formula of Gauss and Legendre:

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z). \tag{2.14}$$

We have

$$\Gamma\left(\frac{1}{12}-\frac{p}{2}\right)\Gamma\left(\frac{7}{12}-\frac{p}{2}\right) = 2^{\frac{5}{6}+p}\sqrt{\pi}\Gamma\left(\frac{1}{6}-p\right),$$

$$\Gamma\left(\frac{5}{12}-\frac{p}{2}\right)\Gamma\left(\frac{11}{12}-\frac{p}{2}\right) = 2^{\frac{1}{6}+p}\sqrt{\pi}\Gamma\left(\frac{5}{6}-p\right).$$

Therefore we have

$$2EF = -4e^{i\pi(-\frac{2}{3}+p)}\frac{\Gamma(1-p)^2}{2^{1+2p}\Gamma\left(\frac{1}{6}-p\right)\Gamma\left(\frac{5}{6}-p\right)}.$$

Recall the multiplication formula of the third degree:

$$\Gamma(z)\Gamma\left(z+\frac{1}{3}\right)\Gamma\left(z+\frac{2}{3}\right) = \frac{2\sqrt{3}\pi}{3^{3z}}\Gamma(3z). \tag{2.15}$$

We have

$$\Gamma\left(\frac{1}{6}-p\right)\Gamma\left(\frac{1}{2}-p\right)\Gamma\left(\frac{5}{6}-p\right) = 2\cdot 3^{3p}\pi\Gamma\left(\frac{1}{2}-3p\right).$$

We use (2.14) again and get

$$\Gamma\left(\frac{1}{2}-p\right)\Gamma(1-p) = 2^{2p}\sqrt{\pi}\Gamma(1-2p).$$

Hence we obtain

$$2EF = -\frac{1}{3^{3p}\sqrt{\pi}}e^{i\pi(-\frac{2}{3}+p)}\frac{\Gamma(1-p)\Gamma(1-2p)}{\Gamma\left(\frac{1}{2}-3p\right)}. \tag{2.16}$$

In the same way,

$$u_6(s)^2 = 2GHu_3(s)u_4(s) + (\text{holomorphic at } s=1).$$

And we get

$$2GH = -\frac{3^{3p}}{\sqrt{\pi}}e^{i\pi(-\frac{2}{3}-p)}\frac{\Gamma(1+p)\Gamma(1-2p)}{\Gamma\left(\frac{1}{2}+3p\right)}. \tag{2.17}$$

We will consider $\phi_{\infty}^{(1)}(s)$. By (2.13) we have

$$u_5(s)u_6(s) = (EH+FG)u_3(s)u_4(s) + (\text{holomorphic at } s=1).$$

Since

$$\Gamma\left(\frac{11}{12}-\frac{p}{2}\right)\Gamma\left(\frac{1}{12}+\frac{p}{2}\right) = \frac{\pi}{\sin\left(\frac{\pi}{12}+\frac{p\pi}{2}\right)},$$

$$\Gamma\left(\frac{7}{12}-\frac{p}{2}\right)\Gamma\left(\frac{5}{12}+\frac{p}{2}\right) = \frac{\pi}{\sin\left(\frac{5\pi}{12}+\frac{p\pi}{2}\right)},$$

we have

$$\begin{aligned} EH+FG &= -\frac{2}{\pi}e^{-\frac{2}{3}i\pi}\Gamma(1-p)\Gamma(1+p) \\ &\quad\cdot\left(\sin\left(\frac{\pi}{12}+\frac{p\pi}{2}\right)\sin\left(\frac{5\pi}{12}+\frac{p\pi}{2}\right)+\sin\left(\frac{11\pi}{12}+\frac{p\pi}{2}\right)\sin\left(\frac{7\pi}{12}+\frac{p\pi}{2}\right)\right) \\ &= -\frac{2}{\pi}e^{-\frac{2}{3}i\pi}\Gamma(1-p)\Gamma(1+p)\cos\frac{\pi}{3} \\ &= -\frac{1}{\pi}e^{-\frac{2}{3}i\pi}\Gamma(1-p)\Gamma(1+p). \end{aligned} \tag{2.18}$$

By (2.16), (2.17) and (2.18) we obtain the proposition. ∎

3. Connection formula around infinity

In this section we will calculate the Stokes multipliers around infinity. At first we will see Stokes regions on (2.1), due to Voros ([V]). The Borel transform $\phi_B(\xi, q)$ of the WKB solution has three branch points

$$\xi_j(q) = -\frac{3}{n+3}\omega^j q^{\frac{3}{n+3}},$$

for $j = -1, 0, 1$. The Stokes line is the curve in q-space defined by the equation

$$\Im\xi_j(q) = \Im\xi_k(q) \tag{3.1}$$

for $j \neq k$. By (3.1), we have

$$L_h : \arg q = \frac{(2h+1)\pi}{2(n+3)}. \qquad (h = 0, 1, 2, \cdots, 2n+5)$$

This definition is different from the notation used in introduction.

We set

$$S_h : \left\{ q; -\frac{(2h-1)\pi}{2(n+3)} < \arg q < \frac{(2h+1)\pi}{2(n+3)} \right\},$$

and calculate the Stokes multipliers from S_0 to S_1. We will denote $\psi_k^{(j)}(q, x)$ is the Laplace transform of $\psi_B^{(j)}(\xi, q)$ when $q \in S_k$.

On L_0 we have

$$\Im\xi_0(q) = \Im\xi_1(q), \qquad \Re\xi_0(q) < \Re\xi_1(q).$$

Therefore the Laplace transform

$$\psi^{(0)}(q, x) = x\int_{\xi_0}^{\infty} e^{-x\xi}\psi_B^{(0)}(\xi, q)d\xi$$

is changed when q moves across the Stokes line L_0, while $\psi^{(1)}(q, x)$ and $\psi^{(-1)}(q, x)$ are not changed.

In [V], Voros showed that the connection formula across the Stokes line is

$$\psi_1^{(1)}(q, x) = \psi_0^{(0)}(q, x) + 2\Delta\psi_0^{(1)}(q, x).$$

Here the complex number Δ is defined by

$$\psi_B^{(0)}(\xi, q) = \Delta\psi_B^{(1)}(\xi, q) + (\text{holomorphic at } \xi = \xi_1(q)),$$

near $\xi = \xi_1(q)$.

PROPOSITION 3.1. *The connection formula from S_0 to S_1 is the followings:*

$$\begin{aligned}
\psi_1^{(0)}(q,x) &= \psi_0^{(0)}(q,x) + \omega^2\left(4\cos^2\frac{\pi}{n+3} - 1\right)\psi_0^{(1)}(q,x),\\
\psi_1^{(1)}(q,x) &= \psi_0^{(1)}(q,x),\\
\psi_1^{(-1)}(q,x) &= \psi_0^{(-1)}(q,x).
\end{aligned}$$

PROOF: We should know the behavior of $\psi_B^{(0)}(\xi,q)$ at $\xi=\xi_1(q)$. By proposition 2.1 the behavior of

$$h^{(0)}(t) = 2\sqrt{\frac{p}{\pi}}\phi_1^{(1)}\left(-\left(\frac{t}{3p}\right)^3\right)$$

at $t=-3p\omega$ is

$$\begin{aligned}
\Delta &= 2\sqrt{\frac{p}{\pi}}\tilde{\phi}_1^{(1)}\left(-\left(\frac{t}{3p}\right)^3\right)\Bigg|_{t=-3p\omega}\\
&= 2\sqrt{\frac{p}{\pi}}\left(2\cos^2\pi p - \frac{1}{2}\right)\omega\phi_1^{(1)}\left(-\left(\frac{t}{3p}\right)^3\right)\Bigg|_{t=-3p\omega}\\
&= \omega^2\left(2\cos^2\pi p - \frac{1}{2}\right)h^{(1)}(t)\Big|_{t=-3p\omega}.
\end{aligned}$$

Therefore $\Delta = \omega^2\left(2\cos^2\pi p - \frac{1}{2}\right)$. ∎

We will calculate the Stokes multiplier when q goes across L_1. On L_1 we have

$$\Im\xi_2(q) = \Im\xi_1(q), \qquad \Re\xi_2(q) < \Re\xi_1(q).$$

Therefore $\xi^{(0)}(q,x)$ and $\xi^{(1)}(q,x)$ are not changed and

$$\psi_2^{(-1)}(q,x) = \psi_1^{(-1)}(q,x) + 2\Delta'\psi_1^{(1)}(q,x).$$

Here the complex number Δ' is defined by

$$\psi_B^{(2)}(\xi,q) = \Delta'\psi_B^{(1)}(\xi,q) + (\text{holomorphic at } \xi=\xi_1(q)),$$

near $\xi=\xi_1(q)$.

PROPOSITION 3.2. *The connection formula from S_1 to S_2 is the followings:*

$$\begin{aligned}
\psi_2^{(0)}(q,x) &= \psi_1^{(0)}(q,x),\\
\psi_2^{(1)}(q,x) &= \psi_1^{(1)}(q,x),\\
\psi_2^{(-1)}(q,x) &= \psi_1^{(-1)}(q,x) + \omega\left(4\cos^2\frac{\pi}{n+3} - 1\right)\psi_1^{(1)}(q,x).
\end{aligned}$$

PROOF: The proof is the same as the proof of proposition 3.1 except that we take $\tilde{\tilde{\phi}}_1^{(1)}$ instead of $\tilde{\phi}_1^{(1)}$. ∎

Since

$$\psi^{(j)}\left(e^{2pi\pi}q,x\right)=e^{i\pi\left(2p-\frac{2}{3}\right)}\psi^{(j+1)}(q,x),$$

the other case is deduced from proposition 3.1 and 3.2.

THEOREM 3.3. *When q goes across L_h, we have the connection formulae between $\psi_h^{(j)}(q,x)$ and $\psi_{h+1}^{(j)}(q,x)$:*

$$\begin{aligned}
\psi_{6h+1}^{(0)}(q,x)&=\psi_{6h}^{(0)}(q,x)+\omega^2\Delta\psi_{6h}^{(1)}(q,x),\\
\psi_{6h+2}^{(-1)}(q,x)&=\psi_{6h+1}^{(-1)}(q,x)+\omega\Delta\psi_{6h+1}^{(1)}(q,x),\\
\psi_{6h+3}^{(-1)}(q,x)&=\psi_{6h+2}^{(-1)}(q,x)+\omega^2\Delta\psi_{6h+2}^{(0)}(q,x),\\
\psi_{6h+4}^{(1)}(q,x)&=\psi_{6h+3}^{(1)}(q,x)+\omega\Delta\psi_{6h+3}^{(0)}(q,x),\\
\psi_{6h+5}^{(1)}(q,x)&=\psi_{6h+4}^{(1)}(q,x)+\omega^2\Delta\psi_{6h+4}^{(-1)}(q,x),\\
\psi_{6h+6}^{(0)}(q,x)&=\psi_{6h+5}^{(0)}(q,x)+\omega\Delta\psi_{6h+5}^{(-1)}(q,x),
\end{aligned}$$

where $\Delta=4\cos^2\dfrac{\pi}{n+3}-1$. In the other case Stokes multipliers are trivial:

$$\psi_{h+1}^{(j)}(q,x)=\psi_h^{(j)}(q,x).$$

4. Connection formula from zero to infinity

(2.1) has solutions

$$y_k(q,x) = x^{\frac{3k}{n+3}} q^k \sum_{m=0}^{\infty} \frac{p^{3m}}{\prod_{s=0}^{2} \Gamma(1+(k-s)p+m)} x^{3m} q^{(n+3)m}, \quad (k=0,1,2) \tag{4.1}$$

near $q=0$. (4.1) converge for $|q|<\infty$. In this section we study connection formula between $y_k(q,x)$ and $\psi(q,x)$. This connection formula gives asymptotic expansions of $y_k(q,x)$, when q is in the large.

The independent solutions of (2.6) near $t=\infty$ are given by the following:
(4.2)

$$\phi_{\infty}^{(k)}\left(-\left(\frac{t}{3p}\right)^3\right) = (3p)^{\frac{1}{2}+3(k-1)p} e^{\pi i\left(-\frac{1}{6}+(1-k)p\right)} (-t)^{-\frac{1}{2}-3(k-1)p} \sum_{m=0}^{\infty} c_m^{(k)} \left(-\frac{3p}{t}\right)^{3m},$$

for $k=0,1,2$, Here

$$c_m^{(k)} = \frac{\left(\frac{1}{6}+(k-1)p\right)_m \left(\frac{1}{2}+(k-1)p\right)_m \left(\frac{5}{6}+(k-1)p\right)_m}{(1+kp)_m (1+(k-1)p)_m (1+(k-2)p)_m}.$$

In (4.2), we take a sector

$$-\frac{2}{3}\pi < \arg(-t) < 0,$$

which corresponds to the sector

$$-\pi < \arg(-s) < \pi,$$

by the transform

$$-s = e^{\pi i} \left(\frac{-t}{3p}\right)^3.$$

The Borel transform $\psi_B(\xi,q)$ is a sum of the following functions near $\xi=\infty$:

$$\psi_{\infty}^{(k)}(\xi,q) = q^{1-\frac{p}{6}} \phi_{\infty}^{(k)}\left(\xi q^{-\frac{1}{3p}}\right). \tag{4.3}$$

We will take Laplace transform of (4.3). Let C be a curve which starts from $+\infty$, turns around $\xi_0(q)$, $\xi_1(q)$, $\xi_{-1}(q)$ counter-clockwise and returns to $+\infty$. And Let C_j $(j=0,1,2)$ be a curve which starts from $+\infty$, turns around $\xi_j(q)$ counter-clockwise and returns to $+\infty$. Consider the Laplace integral

$$y_{\infty}^{(k)} = x\int_C e^{-x\xi} \psi_{\infty}^{(k)}(\xi,q) d\xi.$$

Then we have

$$y_\infty^{(k)} = x \sum_{j=-1}^{1} \Delta_j^{(k)} \int_{C_j} e^{-x\xi} \psi_B^{(j)}(\xi, q) d\xi,$$

where $\Delta_j^{(k)}$ is the discontinuity of $\psi_\infty^{(k)}$ at $\xi = \xi_j(q)$. Since $\psi_B^{(j)}(\xi, q)$ has a singularity of square root type at $\xi = \xi_j(q)$, we have

$$\begin{aligned} y_\infty^{(k)} &= x \sum_{j=-1}^{1} 2\Delta_j^{(k)} \int_{\xi_j(q)}^{\infty} e^{-x\xi} \psi_B^{(j)}(\xi, q) d\xi, \\ &= \sum_{j=-1}^{1} 2\Delta_j^{(k)} \psi^{(j)}(q, x). \end{aligned} \tag{4.4}$$

In the following, we set

$$r_k = (3p)^{\frac{1}{2}+3(k-1)p} e^{\pi i\left(-\frac{1}{6}+(1-k)p\right)}. \quad (k = 0, 1, 2)$$

PROPOSITION 4.1. *we have*

$$y_\infty^{(k)}(q, x) = 2\pi e^{\pi i\left(\frac{1}{3}+(1-k)p\right)} (3p)^{\frac{1}{2}+3(k-1)p} x^{\frac{1}{2}-3p} \frac{\prod_{s=0}^{2} \Gamma(1 + (k-s)p)}{\Gamma\left(\frac{1}{2} + 3(k-1)p\right)} y_k(q, x).$$

PROOF: We can develop $\psi_\infty^{(k)}(\xi)$ near $\xi = -\infty$ as follows:

$$\psi_\infty^{(k)}(\xi) = r_k q^k \sum_{m=0}^{\infty} (3p)^{3m} c_m^{(k)} q^{\frac{m}{p}} (-\xi)^{-\frac{1}{2}-3(k-1)p-3m}.$$

Since

$$\int_C e^{-x\xi}(-\xi)^\alpha d\xi = \frac{2\pi i}{\Gamma(-\alpha)} x^{-1-\alpha},$$

we have

$$y_\infty^{(k)} = 2\pi i r_k x^{\frac{1}{2}+3(k-1)p} q^k \sum_{m=0}^{\infty} \frac{(3p)^{3m} c_m^{(k)}}{\Gamma\left(\frac{1}{2} + 3(k-1)p + 3n\right)} x^{+3m} q^{\frac{m}{p}}.$$

By (2.15) we get

$$\begin{aligned} c_m^{(k)} &= \frac{\prod_{s=0}^{2} \Gamma\left(\frac{1}{6} + (k-1)p + m + \frac{s}{3}\right)}{\prod_{s=0}^{2} \Gamma\left(\frac{1}{6} + (k-1)p + \frac{s}{3}\right)} \frac{\prod_{s=0}^{2} \Gamma(1 + (k-s)p)}{\prod_{s=0}^{2} \Gamma(1 + (k-s)p + m)} \\ &= \frac{\Gamma\left(\frac{1}{2} + 3(k-1)p + 3m\right)}{3^{3m} \Gamma\left(\frac{1}{2} + 3(k-1)p\right)} \frac{\prod_{s=0}^{2} \Gamma(1 + (k-s)p)}{\prod_{s=0}^{2} \Gamma(1 + (k-s)p + m)}. \end{aligned}$$

Therefore we obtain that

$$
\begin{aligned}
y_\infty^{(k)}(q,x) &= \frac{2\pi i r_k}{\Gamma\left(\frac{1}{2}+3(k-1)p\right)} x^{\frac{1}{2}+3(k-1)p} q^k q \sum_{m=0}^{\infty} \frac{p^{3m}\prod_{s=0}^{2}\Gamma(1+(k-s)p)}{\prod_{s=0}^{2}\Gamma(1+(k-s)p+m)} x^{3m} q^{\frac{m}{p}} \\
&= \frac{2\pi i r_k x^{\frac{1}{2}-3p} q^k \prod_{s=0}^{2}\Gamma(1+(k-s)p)}{\Gamma\left(\frac{1}{2}+3(k-1)p\right)} y^{(k)}(q,x) \\
&= 2\pi e^{\pi i\left(\frac{1}{3}+(1-k)p\right)} (3p)^{\frac{1}{2}+3(k-1)p} x^{\frac{1}{2}-3p} \frac{\prod_{s=0}^{2}\Gamma(1+(k-s)p)}{\Gamma\left(\frac{1}{2}+3(k-1)p\right)} y_k(q,x).
\end{aligned}
$$

∎

THEOREM 4.2. *$y_j(q,x)$ has an asymptotic expansion in S_0 in the form*

$$
y_k(q,x) = \frac{p^{3(1-k)p-1}}{2\sqrt{3}\pi} x^{-\frac{1}{2}+3p} \sum_{j=-1}^{1} e^{-j\pi i\left(\frac{2}{3}+2(k-1)p\right)} \psi^{(j)}(q,x).
$$

PROOF: We will consider the behavior of $\phi_\infty^{(k)}(t)$ at $t=-3p\omega^j$ for $j=-1,0,1$. We assume that $\phi_\infty^{(k)}(t)$ has the form

$$
d^{(k)}(t+3p)^{\frac{1}{2}}(1+O(t+3p)) + (\text{holomorphic at } t=-3p)
$$

near $t=-3p$. Since

$$
\phi_\infty^{(k)}(t) = e^{\pi i\left(\frac{1}{3}+2(k-1)p\right)} \phi_\infty^{(k)}(\omega t),
$$

$\phi_\infty^{(k)}(t)$ has the form

$$
\begin{aligned}
&e^{\pi i\left(\frac{1}{3}+2(k-1)p\right)} d^{(k)}(\omega t+3p)^{\frac{1}{2}}(1+O(t+3p\omega^{-1})) + (\text{holomorphic at } t=-3p\omega^{-1}) \\
=&e^{\pi i\left(\frac{2}{3}+2(k-1)p\right)} d^{(k)}(t+3p\omega^{-1})^{\frac{1}{2}}(1+O(t+3p\omega^{-1})) + (\text{holomorphic at } t=-3p\omega^{-1})
\end{aligned}
$$

near $t=-3p\omega^{-1}$.

In the same way $\phi_\infty^{(k)}(t)$ has the form

$$
e^{\pi i\left(-\frac{2}{3}-2(k-1)p\right)} d^{(k)}(t+3p\omega)^{\frac{1}{2}}(1+O(t+3p\omega)) + (\text{holomorphic at } t=-3p\omega)
$$

near $t=-3p\omega$. Therefore we should know only the discontinuity at $t=-3p$.

We will take a path from infinity to $t=-3p$ in the sector $-\frac{2}{3}\pi < \arg(-t) < 0$, In s-space, this path goes from infinity to $s=1$ in the lower space. By proposition 2.2 $\phi_\infty^{(k)}(t)$ has the form

$$
\begin{aligned}
&-\frac{3^{3(k-1)p}}{\sqrt{\pi}} e^{i\pi\left(-\frac{2}{3}+(1-k)p\right)} \frac{\prod_{s=0}^{2}\Gamma(1+(k-s)p)}{\Gamma\left(\frac{1}{2}+3(k-1)p\right)} \phi_1^{(1)}\left(-\left(\frac{t}{3p}\right)^3\right) \\
&\qquad + (\text{holomorphic at } t=-3p).
\end{aligned} \tag{4.5}
$$

By (2.8) the singular part of (4.5) is

$$-\frac{3^{3(k-1)p}}{2\sqrt{p}}e^{i\pi\left(-\frac{2}{3}+(1-k)p\right)}\frac{\prod_{s=0}^{2}\Gamma\left(1+(k-s)p\right)}{\Gamma\left(\frac{1}{2}+3(k-1)p\right)}h^{(0)}(t).$$

Therefore

$$\Delta_0^{(k)}=-\frac{3^{3(k-1)p}}{2\sqrt{p}}e^{i\pi\left(-\frac{2}{3}+(1-k)p\right)}\frac{\prod_{s=0}^{2}\Gamma\left(1+(k-s)p\right)}{\Gamma\left(\frac{1}{2}+3(k-1)p\right)} \tag{4.6}$$

in (4.4). By proposition 4.1, we obtain the proposition 4.2. ∎

We considered x is a large parameter, and q is a finite number. From now on, we take a variable

$$z=x^{3p}q, \tag{4.7}$$

and consider z in the large. By (4.7), the equation (2.1) is

$$\frac{d^3}{dz^3}\psi-z^n\psi=0. \tag{4.8}$$

The local solution (4.1) is

$$\tilde{y}_k(z)=z^k\sum_{m=0}^{\infty}\frac{p^{3m}}{\prod_{s=0}^{2}\Gamma\left(1+(k-s)p+m\right)}z^{(n+3)m}.$$

And the WKB solution (2.4) is

$$\psi^{(j)}(z)=x^{\frac{1}{2}-3p}z^{-\frac{n}{3}}\exp(3p\omega^j\tilde{q}^{\frac{1}{3p}})\sum_{k\geq 0}t_k z^{\frac{1}{3p}}.$$

We set

$$\tilde{\psi}^{(j)}(z)=\tilde{q}^{-\frac{n}{3}}\exp(3p\omega^j z^{\frac{1}{3p}})\sum_{k\geq 0}t_k z^{\frac{1}{3p}}.$$

By theorem 4.2 we have asymptotic expansion of solutions of (4.8).

THEOREM 4.3. *In the sector* $|\arg z|<\dfrac{\pi}{2(n+3)}$, *we have an asymptotic expansion*

$$\tilde{y}_k(z)=\frac{(n+1)^{\frac{3(k-1)}{3+n}+1}}{2\sqrt{3}\pi}\sum_{j=-1}^{1}e^{-j\pi i\left(\frac{2}{3}+\frac{2(k-1)}{n+3}\right)}\tilde{\psi}^{(j)}(z).$$

REFERENCES

[AKT] T.Aoki, T.Kawai and Y.Takei, The Bender-Wu analysis and the Voros theory, ICM-90 Satellite Conference Proceedings, Special Functions, Springer-Verlag, 1991, pp.1-29.

[AKT2] T.Aoki, T.Kawai and Y.Takei, New turning points in the exact WKB analysis for higher-order ordinary differential equations, Proc. "Algebraic Analysis and Singular perturbations", 1994.

[[BNR] H.L.Berk, W.M.Nevins and K.V.Roberts, New Stokes' line in WKB theory, J. Math. Phys., Math. Soc., **23**(1982), 988-1002.

[S] H.Scheffé, Asymptotic solutions of certain linear differential equations in which the coefficients of the parameter may have a zero, Trans. Amer. Math. Soc., **40**(1936), 127-154.

[SNH] H.J.Silverstone, S. Nakai and J.G.Harris, Observations on the summability of confluent hypergeometric functions and on semiclassical quantum mechanics, Phys. Rev. **32**(1985), 1341-1345.

[T] H.L.Turittin, Stokes multipliers for asymptotic solutions of a certain differential equation, Trans. Amer. Math. Soc., **68**(1950), 304-329

[V] A.Voros, The return of the quartic oscillator. The complex WKB method. Ann. Inst. Henri Poincare, **39** (1983), 211-338.